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LETTER TO THE EDITOR

Scattering by a corrugated random surface with fractal slope

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Abstract. The single-fold statistics of rays emanating from an infinite, corrugated, gaussian surface with fractal slope are investigated. Low moments of the ray-density fluctuation distribution are evaluated as a function of fractal dimension, D. It is shown that in the Brownian case, $D = 1.5$, the distribution is exactly negative exponential, corresponding to K-distributed intensity fluctuations in a coherent scattering configuration.

Over the last three decades a great deal of effort has been devoted to understanding *scintillation* or amplitude fluctuations in waves scattered by random media such as rough surfaces, thin diffusing layers or extended regions of turbulence. There are many familar phenomena of this kind, for example the twinkling of starlight and the fading of radio waves, and their practical importance both as a source of information and in a noise context has long been recognised.

The simplest mathematical model known to generate scintillation effects is the phase-changing screen, which introduces spatially random distortions into an incident planar wavefront. Amplitude fluctuations then develop during the course of free propagation beyond the scattering plane (see, for example, Zardecki (1978) and references therein). This model, which may be used to represent scattering by an ideal rough surface, has been extensively investigated for the case when the surface height fluctuations (wavefront distortions) constitute a joint-gaussian process and is then completed by specification of the height fluctuation spectrum. Two extreme surface types have been recognised. Mathematically, these can be identified as (1) fractal surfaces which are continuous but not differentiable, with power law spectra (Berry 1979), and (2) smoothly varying surfaces which are differentiable to all orders and have gaussian-like spectral properties (Berry 1978). Type (1) surfaces generate only diffraction and interference effects, whilst type (2) surfaces also generate geometrical optics effects associated with rays or normals to the initial scattered wavefront. In the short-wavelength limit the latter dominate the statistical properties of the scattered intensity through the presence of singularities or catastrophes in the wave field. Indeed, the ray-density fluctuations diverge and the smoothing effects of diffraction have to be included in order to calculate the asymptotic behaviour of the intensity statistics in this limit (Berry 1976, 1978).

In this Letter the scattering properties of an intermediate surface, type (3), are investigated. It will be assumed that the surface height is continuous and differentiable but its *slope* is a fractal. The concept of rays is valid for this model, but in the absence of higher surface derivatives no geometrical catastrophes occur in the propagating wave

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field. This means that the short-wavelength limit can be examined without recourse to diffraction smoothing, i.e. by consideration of fluctuations in the density of rays only, and results in significantly simpler calculations than the type (2) surface discussed above. Type (3) scatterers, which may be thought of as having a facet-like structure, have been investigated recently in connection with the propagation of radio waves through the ionosphere (Rumsey 1975, Rino 1979), and a thorough discussion of the subject is given by Uscinski *et al* (1981). The object of the present work, however, is to draw attention to the close similarity between statistical predictions obtained by analytical solution of the scattering problem for model type (3) in the geometrical optics limit and the basic assumptions of a more empirical approach which has successfully explained certain experimental results (Jakeman and Pusey 1978, Jakeman 1980a, b).

For simplicity, calculations in this Letter will be confined to a one-dimensional (corrugated) model of infinite lateral extent, i.e. in a Fresnel region configuration. A starting point for the calculations is the functional

$$R=\frac{1}{z_0}\int_{-\infty}^{\infty}\delta(m(x)-x/z_0)\,\mathrm{d}x. \tag{1}$$

This defines the ray density at a distant point z_0 beyond a wavefront in the $z=0$ plane which is corrugated in the x direction and has local random slope $m(x)$. Evidently

$$\langle R\rangle=1. \tag{2}$$

Using the results of gaussian noise theory, the Nth *normalised* moment of R may be written in the form

$$\langle R^N\rangle=\frac{1}{(2\pi z_0)^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\ldots\mathrm{d}x_N\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N$$
$$\times\exp\left(\frac{\mathrm{i}}{z_0}\sum_{j=1}^{N}\lambda_jx_j-\tfrac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\lambda_j\lambda_k\langle m(x_j)m(x_k)\rangle\right). \tag{3}$$

The multi-dimensional integral over $\{x_j\}$ can be expressed as the sum of $N!$ contributions in which these variables are differently ordered. For example, when $N=3$ the six regions $x_1<x_2<x_3$; $x_1<x_3<x_2$; $x_2<x_3<x_1$; $x_2<x_1<x_3$; $x_3<x_1<x_2$; $x_3<x_2<x_1$ span the entire three-dimensional space of integration. Each contribution is invariant under interchange of indices, so that

$$\langle R^N\rangle=\frac{N!}{(2\pi z_0)^N}\int_{-\infty}^{\infty}\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{x_1}^{\infty}\mathrm{d}x_2\int_{x_2}^{\infty}\mathrm{d}x_3\ldots\int_{x_{N-1}}^{\infty}\mathrm{d}x_N$$
$$\times\exp\left(\frac{\mathrm{i}}{z_0}\sum_{j=1}^{N}\lambda_jx_j\right)\exp\left(\tfrac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\lambda_j\lambda_k\langle m(x_j)m(x_k)\rangle\right). \tag{4}$$

Assuming that $m(x)$ is a stationary random function of x, the transformation $x_j\to x_j+x_1$ for $j\geqslant 2$ allows the x_1 and λ_1 integrals to be performed exactly, to give

$$\langle R^N\rangle=\frac{N!}{(2\pi z_0)^{N-1}}\int_{-\infty}^{\infty}\mathrm{d}\lambda_2\ldots\mathrm{d}\lambda_N\int_0^{\infty}\mathrm{d}x_2\int_{x_2}^{\infty}\mathrm{d}x_3\ldots\int_{x_{N-1}}^{\infty}\mathrm{d}x_N$$
$$\times\exp\left(\frac{\mathrm{i}}{z_0}\sum_{j=2}^{N}\lambda_jx_j\right)\exp\left[\tfrac{1}{2}\sum_{j=2}^{N}\lambda_j\left(\sum_{k=j+1}^{N}\lambda_kS(x_j-x_k)-\sum_{k=2}^{N}\lambda_kS(x_k)\right)\right] \tag{5}$$

where

$$S(x-y)=\langle[m(x)-m(y)]^2\rangle \tag{6}$$

is the slope structure function. It is not difficult to check that the integrals (5) diverge if $S(x)$ is an even integral-powered function of x corresponding to smoothly varying type (2) surfaces. However, finite results are obtained for the fractal model

$$S(x)=L^{2(D-1)}|x|^{2(2-D)}, \qquad 1<D<2, \tag{7}$$

where D is the fractal dimension and L the topothesy of $m(x)$ (Berry 1979).

The second moment can be evaluated for arbitrary D in the allowed range, leading to

$$\langle R^2\rangle=(D-1)^{-1}, \tag{8}$$

whilst the third moment can be expressed in the reduced form

$$\langle R^3\rangle=\frac{3!}{4\pi(D-1)}\int_0^\infty \frac{\mathrm{d}t\{4t^{4-2D}-[t^{4-2D}+1-(t+1)^{4-2D}]^2\}^{1/2}}{t^{4-2D}+t^2+t[t^{4-2D}+1-(t+1)^{4-2D}]}. \tag{9}$$

For the Brownian fractal, $D=1.5$, the integral in equation (9) can be evaluated to give

$$\langle R^3\rangle=3! \qquad (D=1.5). \tag{10}$$

In this latter case the right-hand side of equation (5) can be further reduced by successive transformations of the type

$$\left.\begin{aligned} &x_j\to x_j+x_k \qquad (j\geqslant k+1)\\ &\lambda_k\to\lambda_k-\sum_{j=k+1}^{N}\lambda_j \end{aligned}\right\} \text{ for } k=2,3,4\ldots N-1, \tag{11}$$

leading finally to the result

$$\langle R^N\rangle=N!\left[\frac{1}{2\pi z_0}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_0^\infty \mathrm{d}x\,\exp\left(\frac{\mathrm{i}\lambda x}{z_0}-\tfrac{1}{2}L\lambda^2x\right)\right]^{N-1}=N!. \tag{12}$$

Thus the density of rays from a corrugated gaussian surface with Brownian fractal slope has a negative exponential distribution:

$$P(R)=\exp(-R) \qquad (\text{for } D=1.5). \tag{13}$$

The distribution (13) is the most familiar member of the class of gamma variates (with unit mean):

$$P(X)=X^{\alpha-1}\exp(-X)/\Gamma(\alpha) \tag{14}$$

where

$$\langle X^2\rangle=1+\alpha^{-1} \tag{15}$$

and it is interesting to compare the third moment (9) with the third moment of (14) when the second moments (8) and (15) coincide, i.e. when $\alpha=(D-1)/(2-D)$. Figure 1 shows computed results for various values of D in the allowed range. It is clear that the ray density is close to being gamma distributed for a wide range of second-moment values, but particularly when $1.5<D<2$. This allows the presence of an outer scale or low-frequency cut-off not incorporated into (7) (Uscinski *et al* 1981) to be taken into account in a simple way. The effect of a finite correlation length within the scatterer is

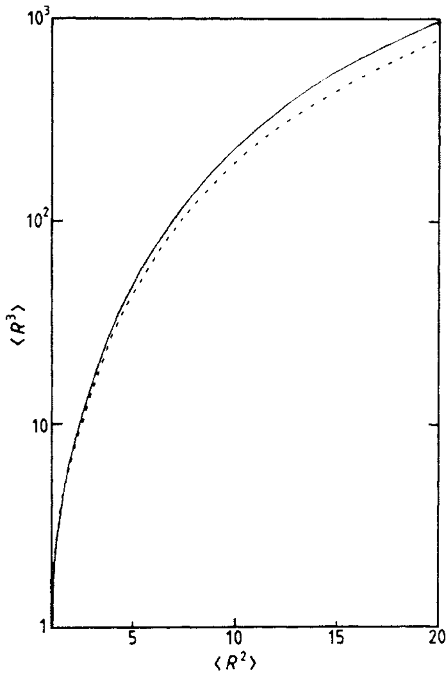

Figure 1. Comparison of the third moment of the distribution of ray-density fluctuations generated by a corrugated surface with fractal slope (——) and the third moment of a gamma distribution with the same variance (- - -). $\langle R \rangle = 1$.

that R will comprise a number of *independent* contributions, each of which will be approximately (or exactly if $D = 1.5$) gamma distributed with parameter α, say. By virtue of the infinitely divisible nature of the class (14), the sum of M such contributions will also be gamma distributed but with increased index $M\alpha$ corresponding to a reduced degree of fluctuation. Thus the presence of a finite outer scale will not affect the general character of the ray-density fluctuations, but only their magnitude (variance).

The results of the above calculation of ray-density statistics are directly applicable only to incoherent scattering configurations where broadband (e.g. white light) illumination or spatially extended sources are used, or where spatial or temporal averaging of interference effects occurs in the detection system. In coherent scattering configurations, when the complex wave amplitude can be represented by a two-dimensional random walk, the ray-density fluctuations are analogous to variations in the step number, which modulate the *local* mean intensity of the resultant interference pattern. It is well known that in the large-step-number limit the latter is characterised by a negative exponential distribution of intensity fluctuations (gaussian speckle: see for example Dainty (1975)). Random modulation of the mean of such a distribution according to the class (14) generates K-distributed fluctuations (Jakeman 1980a, b). In particular, for the Brownian case, $D = 1.5$, the following exact result is obtained, assuming unit mean for simplicity:

$$\langle I^N \rangle = N! \langle R^N \rangle = (N!)^2, \tag{16}$$

corresponding to the K distribution

$$P(I) = 2K_0(2\sqrt{I}). \tag{17}$$

Although the above method of incorporating interference effects is somewhat empirical, the results are in agreement with those obtained by a more rigorous treatment of the coherent scattering problem, to be presented elsewhere.

The noteworthy feature of the results derived in this Letter is that whereas number density fluctuations of the type (14) have to date been invoked *ad hoc* to explain the observation of K-distributed scintillation, it appears that certain multi-scale scattering models uniquely *predict* distributions of this kind.

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